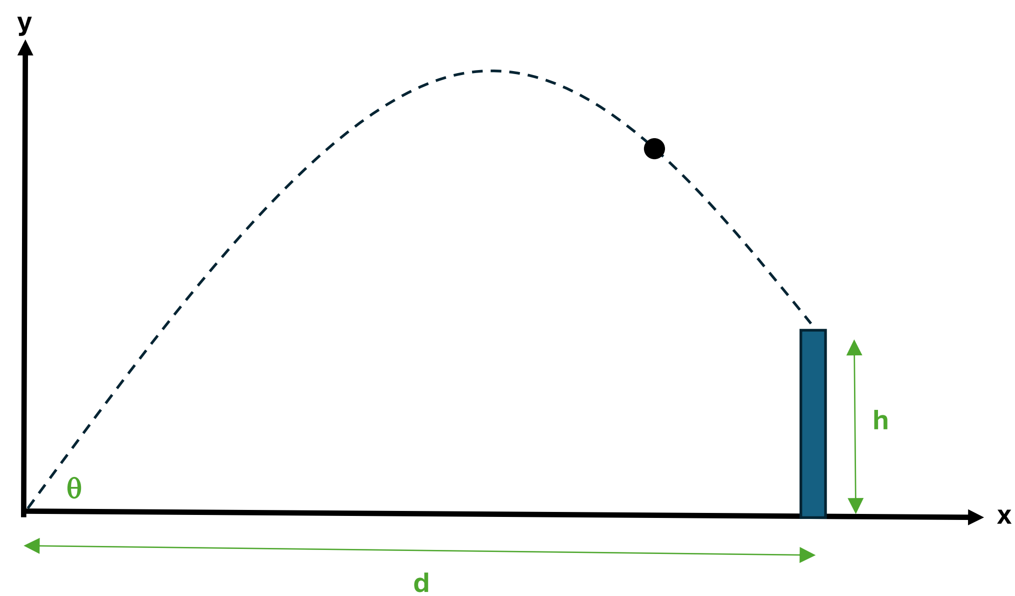
Trajectory Modeling

Air Resistance, Spin Effects, and Moving Platform



Dean Banerjee

Robotics Team Mentor

April 21, 2026

Introduction

This document develops a trajectory model for a ball launched from a robotics system, with the goal of predicting impact location and understanding the key factors that influence accuracy. The analysis begins with the ideal no-air-resistance case to establish a closed-form solution and provide intuition for how launch angle and velocity affect the trajectory. The model is then extended to include aerodynamic effects such as drag and the Magnus force, which are solved numerically. These models are compared against measured data from the robot to evaluate accuracy and identify the dominant physical effects.

The intent is not to produce a perfect physical model, but to create a practical framework that can be used to guide tuning, testing, and system design. Emphasis is placed on understanding which effects matter most and how they influence real-world performance.

Consider the case of a turret mounted on a vehicle that is trying to fire a projectile to hit a target. This involves calculating the proper combination of angle and muzzle velocity of the projectile to do so. In the situation that the vehicle is not moving and in the absence of air resistance, this has an elegant closed-form solution. In the case of no air resistance and the vehicle is moving, then there is still a closed form solution, although it is more involved if the movement contains components that are not purely towards or away from the target. Air resistance is also considered, and this requires numerical methods to solve. This paper treats these four cases:

- Stationary turret and target with no air resistance
- Moving turret or target with no air resistance
- Stationary turret and target case with air resistance
- Moving turret or target with air resistance

Trajectory Calculation No Air Resistance

As an initial start, first define the key necessary parameters as shown in Table 1 and Figure 1.

Parameter	***Symbol***	***Units***
Projectile Initial Velocity	v_0	*m/s*
Projectile Initial Angle	θ	*deg*
Height of Target	h	*m*
Distance to Target	d	*m*
Acceleration Due to Gravity	g = 9.8	*m/s²*

Table 1 *Known Conditions*

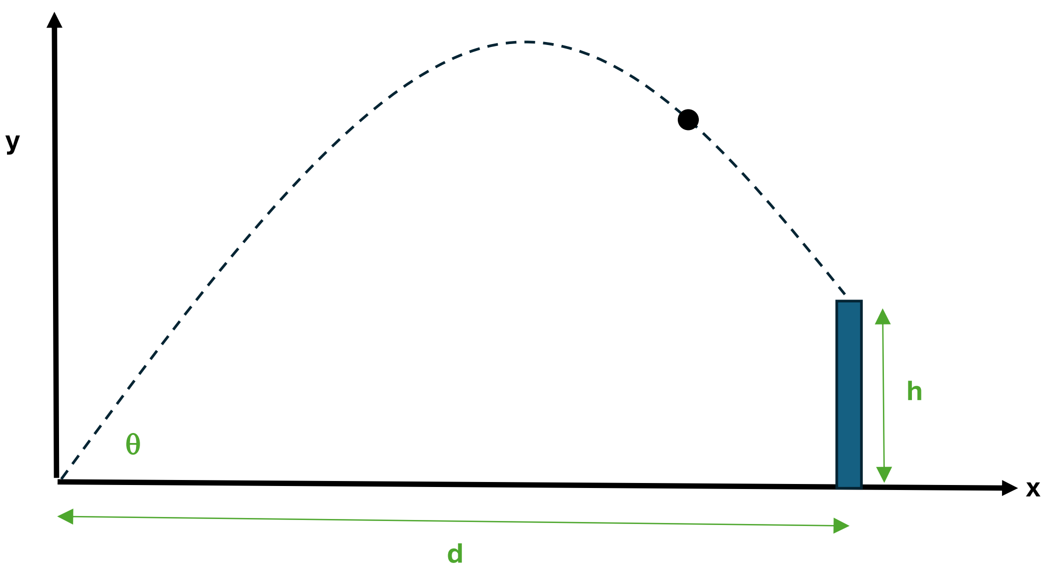


Figure 1 *Trajectory Path*

Calculation of Trajectory Curve

Although the time in the air may not be of interest, it is easiest to use this variable and eliminate it from the equations to find the equations. The distance at any time t can be found as:

$$x = v_0 \cdot t \cdot cos\theta \tag{1}$$

From this, the corresponding time can be solved for.

$$t = \frac{x}{v_0 \cdot cos\theta} \tag{2}$$

The height at time t can be found as:

$$y = v_0 \cdot t \cdot sin\theta - \frac{1}{2} \cdot g \cdot t^2 \tag{3}$$

Combining (1), (2), and (3) yields the general trajectory curve:

$$y = x \cdot tan\theta - \frac{g \cdot x^2}{2 \cdot {v_0}^2 \cdot cos^2\theta} \tag{4}$$

Finding Peak Time, Peak Height, and X Coordinate at Peak

The first step is to differentiate (3) to find the time where the vertical velocity is zero.

$$\frac{dy}{dt} = 0 = v_0 \cdot sin\theta - g \cdot t \tag{5}$$

$$t_{peak} = \frac{v_0 \cdot sin\theta}{g} \tag{6}$$

Now that this peak time is known, the peak height can be found by substituting (6) into (3)

$$y_{peak} = v_0 \cdot \left(\frac{v_0 \cdot sin\theta}{g}\right) \cdot sin\theta - \frac{1}{2} \cdot g \cdot \left(\frac{v_0 \cdot sin\theta}{g}\right)^2 = \frac{(v_0 \cdot sin\theta)^2}{2 \cdot g} \tag{7}$$

The x coordinate can be found by substituting (6) into (1).

$$x_{peak} = v_0 \cdot \left(\frac{v_0 \cdot sin\theta}{g}\right) \cdot cos\theta = \frac{{v_0}^2 \cdot sin\theta \cdot cos\theta}{g} \tag{8}$$

Finding the Time to Hit Ground and Range

To find the time to hit ground, set y=0 into (3).

$$0 = v_0 \cdot t \cdot sin\theta \; - \frac{1}{2} \cdot g \cdot t^2 \tag{9}$$

$$t_{Range} = \frac{2 \cdot v_0 \cdot sin\theta}{g} \tag{10}$$

To find the range, substitute (10) into (1)

$$x_{Range} = v_0 \cdot \left(\frac{2 \cdot v_0 \cdot sin\theta}{g}\right) \cdot cos\theta \; = \; \frac{2 \cdot {v_0}^2 \cdot sin\theta \cdot cos\theta}{g} \tag{11}$$

Calculation of Angle and Velocity to Hit Target

To find the combination of angle and velocity, realize that (d,h) is a point on the curve in (4).

$$h = d \cdot tan\theta \; - \; \frac{g \cdot d^2}{2 \cdot {v_0}^2 \cdot cos^2\theta} \tag{12}$$

If the angle is known, the velocity can easily be solved for using (12).

$$v_0 = \frac{d}{cos\theta} \cdot \sqrt{\frac{g/2}{d \cdot tan\theta \; - h}} \tag{13}$$

If the speed is known, a little more is required. The first step is to remember the identity:

$$1 + tan^2\theta = sec^2\theta = \; \frac{1}{cos^2\theta} \tag{14}$$

Substituting (14) into (12) and rearranging yields the following result:

$$h = d \cdot tan\theta \; - \; \frac{g \cdot d^2 \cdot (1 + tan^2\theta)}{2 \cdot {v_0}^2} \tag{15}$$

This can be restated as follows:

$$a \cdot tan^2\theta + b \cdot tan\theta + c = 0 \tag{16}$$

$$a = g \cdot d^2 \tag{17}$$

$$b = \; -2 \cdot {v_0}^2 \cdot d \tag{18}$$

$$c = 2 \cdot {v_0}^2 \cdot h + g \cdot d^2 \tag{19}$$

The angle can be solved by applying the quadratic equation to (16)

$$\theta_{1,2} = tan^{-1}\left(\frac{-b \pm \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a}\right) \tag{20}$$

Note that (16) leads to 2 solutions. The larger angle we will call the high angle and the smaller angle we will call the low angle. (6), (7), and (8) can be used to find the peak time, peak height, and X coordinate at the peak height. One additional metric would be the time to hit target. This can be found by taking the total distance (d) divided by the x velocity as used in (2).

$$t_{Target} = \frac{d}{v_0 \cdot cos\theta} \tag{21}$$

The slope of projectile when it hits the target can be taken from the derivative of (4) at the point (d,h).

$$m_{1,2} = \frac{dy}{dx} = tan\theta_{1,2} - \frac{g \cdot d}{{v_0}^2 \cdot cos^2\theta_{1,2}} \tag{22}$$

From this slope, the actual angle can be calculated as well

$$\phi_{1,2} = tan^{-1}(m_{1,2}) \tag{23}$$

Appendix A shows an example calculation for the case of a stationary vehicle shooting at a target with no air resistance.

Shooting on the Move with No Air Resistance

This case is split into cases where the motion is purely in line with the target versus in a direction that introduces a component that is also perpendicular to the target. Note that if in fact the vehicle is stationary and the target is moving in a linear fashion, these same equations can be used.

Moving Purely Towards or Away from Target

For a moving launch platform, it is useful to discern between the total angle and velocity and those of the cannon alone. v_0 and θ denote the total launch velocity and angle of the projectile. $\widehat{v_0}$ and $\hat{\theta}$ denote the initial velocity and firing angle of the cannon alone.

$$v_x = \widehat{v_0} \cdot cos\hat{\theta} + v_{Vehicle} \tag{24}$$

$$v_y = \widehat{v_0} \cdot sin\hat{\theta} \tag{25}$$

$$v_0 = \sqrt{{v_x}^2 + {v_y}^2} \tag{26}$$

$$\theta = \text{atan}\left(\frac{v_y}{v_x}\right) \tag{27}$$

$$v_0 = \sqrt{\left(\widehat{v_0} \cdot cos\hat{\theta} + v_{Vehicle}\right)^2 + \left(\widehat{v_0} \cdot sin\hat{\theta}\right)^2} \tag{28}$$

Moving an Arbitrary Direction Toward or Away from the Target No Air Resistance

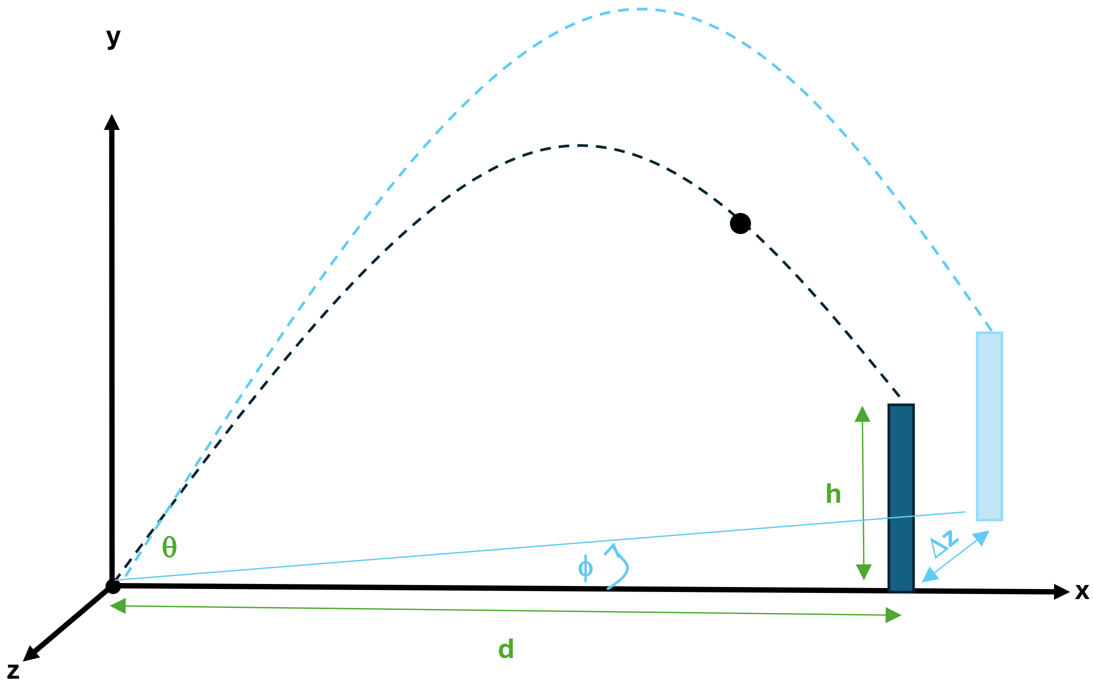


Figure 2 *Trajectory Path*

In the case that there is motion that is not purely towards or away from the target, the first step is to take the turret motion and decompose this motion into a component going towards the target and another going perpendicular to the target. This is why a new z direction is introduced in Figure 2. The starting conditions for the projectile are as follows:

Parameter	Vehicle	Cannon	Total
v_{0x}	$v_m \cos\,\phi_m$	z	$v_0 \cos\,\theta \cos\,\phi + v_m \cos\,\phi_m$
v_{0y}	0	$v_0 \sin\,\theta$	$v_0 \sin\,\theta$
v_{0z}	$v_m \sin\,\phi_m$	$v_0 \cos\,\theta \sin\,\phi$	$v_0 \cos\,\theta \sin\,\phi + v_m \sin\,\phi_m$
Magnitude	v_m	v_0	$\sqrt{v_{0x}^2 + v_{0y}^2 + v_{0z}^2}$
Elevation	0	θ	$\tan^{-1}\left(\frac{v_{0y}}{\sqrt{v_{0x}^2 + v_{0z}^2}}\right)$
Azimuth ϕ	ϕ_m	ϕ	$\tan^{-1}\left(\frac{v_{0z}}{v_{0x}}\right)$

Table 2 *Starting Conditions for the Projectile*

If one keeps the original angle for the stationary case, then the x-velocity will be slightly slower and therefore the projectile will land slightly short of the target in the x-direction. For this reason, the first step in calculating a closed form solution is to solve for the time.

Assuming that the velocity is fixed, a general solution method can be derived by calculating the time the projectile is in the air. For these equations, the velocities are the total velocity which combines the cannon and the moving vehicle. To start, calculate the cannon velocities in the x, y, and z directions

$$t \cdot (v_{0x} + v_m \cdot cos\phi_m) = d \tag{29}$$

$$t \cdot v_{0y} - \frac{1}{2} \cdot g \cdot t^2 = h \tag{30}$$

$$t \cdot v_{0z} + t \cdot v_m \cdot sin\phi_m = 0 \tag{31}$$

These can be solved for the x, y, and z components of the turret velocity.

$$v_{0x} = \frac{d}{t} - v_m \cdot cos\phi_m \tag{32}$$

$$v_{0y} = \frac{h}{t} + \frac{1}{2} \cdot g \cdot t \tag{33}$$

$$v_{0z} = -v_m \cdot sin\phi_m \tag{34}$$

Also, the total launch velocity accounting for both the cannon and vehicle motion is:

$${v_0}^2 = {v_{0x}}^2 + {v_{0y}}^2 + {v_{0z}}^2 \tag{35}$$

Substituting (32), (33), and (34) into (35) yields the following:

$$t^2 \cdot {v_0}^2 = (d - t \cdot v_m \cdot cos\phi_m)^2 + \left(h + \frac{1}{2} \cdot g \cdot t^2\right)^2 + (-t \cdot v_m \cdot sin\phi_m)^2 \tag{36}$$

(36) can be expanded, multiplied through by t^2, and grouped by powers of t to get (37).

$$t^4 + A \cdot t^2 + B \cdot t + C = 0 \tag{37}$$

$$A = \frac{4 \cdot (g \cdot h + {v_m}^2 - {v_0}^2)}{g^2} \tag{38}$$

$$B = -\frac{8 \cdot d \cdot v_m \cdot cos\phi_m}{g^2} \tag{39}$$

$$C = \frac{4 \cdot (d^2 + h^2)}{g^2} \tag{40}$$

This 4th order polynomial can be solved with closed form solution as presented in Appendix B or with numerical methods. Once the time is known, the velocities can be calculated with (32), (33), and (34) and the angles can be found. Appendix C shows a sample calculation using this method.

$$\theta = tan^{-1}\left(\frac{v_{0y}}{\sqrt{{v_{0x}}^2 + {v_{0z}}^2}}\right) \tag{41}$$

$$\phi = tan^{-1}\left(\frac{v_{0z}}{v_{0x}}\right) \tag{42}$$

Trajectory Calculation Accounting for Air Resistance

In the case of air resistance, there is no closed form solution and numerical methods are required. Unlike the case with no resistance, it is different whether the vehicle is moving or the target is moving. Air resistance has three components. The drag force is the head on air resistance. The shear force is more for longer objects. In the case of a ball, the shear force is small and can be disregarded, but it is included for completeness. The Magnus force is the one that accounts for the spin of the ball. These 3 forces are discussed in appendices D, E, and F, respectively. For the purposes of simplification, it is assumed that the axis of the spin for the ball does not change.

Case of a Stationary Vehicle

Step 1: Define the Time Step, Δt

The time step is important for calculations. If it is too large, the accuracy of the model will not be good as numerical methods lose accuracy as the time step gets too coarse. However, if it is too small, then computational time will be too long. For the simulations that were run, it seemed that an appropriate time step that balanced these features was:

$$\Delta t \equiv 0.0002\ s \tag{43}$$

Step 2: Start with Initial Position and Velocity

The initial positions are all known and it is assumed that, by definition, the origin is at the exit point of the barrel.

Parameter	***Symbol/Formula***	***Units***
Initial x Velocity	$v_x(0) = v_0 \cdot cos\theta$	*m/s*
Initial y Velocity	$v_y(0) = v_0 \cdot sin\theta$	*m/s*
Initial x position	x = 0	*m*
Initial y Position	y = 0	*m*
Initial Spin	$\omega(0) = \omega_0$	*rad/s*

Table 3 *Starting Conditions for the Projectile*

Step 3: Calculate the Forces Acting on the Projectile

The four forces acting on the projectile are:

- Gravity
- Aerodynamic Drag Force
- Aerodynamic Shear Force
- Aerodynamic Magnus force

The aerodynamic forces are discussed in detail in appendices D, E, and F. Table 4 shows all the forces acting on the projectile. Note that because these forces are changing between times t and t+Δt, an average is taken.

Force	***Magnitude***	***X component***	***Y Component***
Gravity	$F_g = m \cdot g$	*0*	$-m \cdot g$
Drag	$F_{drag} = \alpha \cdot v^2$	$-\frac{v_x}{v} \cdot F_{drag}$	$-\frac{v_y}{v} \cdot F_{drag}$
Shear	$F_{shear} = \beta \cdot v$	$-\frac{v_x}{v} \cdot F_{shear}$	$-\frac{v_y}{v} \cdot F_{shear}$
Magnus	$F_{magnus} = \gamma \cdot \omega \cdot v$	$-\gamma \cdot \omega \cdot v_y$	$\gamma \cdot \omega \cdot v_x$

Table 4 *Force Components Acting on the Projectile Fired from Stationary Launch Platform*

Table 4 introduces the constants α, β, and γ which can be calculated with the formulae in Table 5 and are numerically evaluated in Appendix G.

Force	***General Formula***	***Formula for a Ball***
α	$\frac{1}{2} \cdot \rho \cdot C_d \cdot A_{Cross}$	$0.26 \cdot \pi \cdot r^2 \cdot \rho$
β	$\mu \cdot A_{Effective}$	$\mu \cdot 6\pi \cdot r$
γ	$\rho \cdot \eta \cdot \pi \cdot r^3$	

Table 5 *Force Coefficient Definitions*

The constants in Table 5 depend on several other values that are presented in Table 6. One thing worthy of pointing out is that the Magnus force coefficient (γ) is the only one that explicitly has the air coupling coefficient (η) stated in its formula. However, the roughness of the air does impact the other force constants, it is just the case that this is absorbed in C_d for the case of the drag coefficient (α) and considered in the derivation of the formula for the shear coefficient (β).

Symbol	***Parameter***	***Typical Value***	***Unit***
ρ	Air density	1.257	kg/m^3
μ	Air viscosity	1.8×10^{-5}	Pa·s
η	Air Coupling Coefficient	0.4	n/a
A_{Cross}	Cross Sectional Area	$\pi \cdot r^2$	m^2
$A_{Effective}$	Effective Area Exposed to Fluid	$6\pi \cdot r$	m^2
C_d	Aerodynamic drag coefficient	0.52	n/a

Table 6 *Air Property Constants*

So at time, t, the force acting on the projectile is as follows:

$$F_x = -\frac{v_x(t)}{v(t)} \cdot F_{drag} - \frac{v_x(t)}{v(t)} \cdot F_{shear} - \gamma \cdot \omega \cdot v_y \tag{44}$$

$$F_y = -\frac{v_y(t)}{v(t)} \cdot F_{drag} - \frac{v_y(t)}{v(t)} \cdot F_{shear} + \gamma \cdot \omega \cdot v_x - m \cdot g \tag{45}$$

Step 4: Calculate the New Velocity and Position

The velocity at time t+Δt can be calculated using Newton's second law.

$$F = m \cdot a \qquad \rightarrow \qquad a = \frac{F}{m} \tag{46}$$

From this, the new velocity can be calculated

$$v_x(t + \Delta t) = v_x(t) + \frac{F_x(t)}{m} \cdot \Delta t \tag{47}$$

$$v_y(t + \Delta t) = v_y(t) + \frac{F_y(t)}{m} \cdot \Delta t \tag{48}$$

Once the new velocity is known, then the new position can be calculated as well.

$$x(t + \Delta t) = x(t) + \frac{v_x(t) + v_x(t + \Delta t)}{2} \cdot \Delta t \tag{49}$$

$$y(t + \Delta t) = y(t) + \frac{v_y(t) + v_y(t + \Delta t)}{2} \cdot \Delta t \tag{50}$$

Step 5: Calculate the New Spin

The spin on the projectile at the new time can be calculated from the initial spin as follows:

$$\omega(t + \Delta t) = \omega_0 \cdot e^{-(t+ \Delta t)/\tau} \tag{51}$$

So now that the new position is calculated, steps 3,4, and 5 can be repeated to calculate the trajectory of the projectile. Appendix J shows a sample calculation.

Moving an Arbitrary Direction Accounting for Air Resistance

If the vehicle is moving purely towards or away from the target, then the initial launch velocity of the projectile can be modified and the previous method for a stationary launch can be applied. However, if there is motion that is in the z direction, this adds some more complication especially for the Magnus force. The first step is to calculate the initial conditions at time of launch.

Component	Initial Velocity	Initial Spin
Magnitude	$\sqrt{v_{0x}^2 + v_{0y}^2 + v_{0z}^2}$	ω_0
X component	$v_{0x} = v_0 \cos\theta \cos\phi + v_m \cos\phi_m$	$\omega_{0x} = -\omega_0 \cdot sin(\phi + \phi_m)$
Y Component	$v_{0y} = v_0 \sin\theta$	$\omega_{0y} = 0$
Z Component	$v_{0z} = v_0 \cos\theta \sin\phi + v_m \sin\phi_m$	$\omega_{0z} = \omega_0 \cdot cos(\phi + \phi_m)$

Table 7 *Initial Conditions for Air Resistance on Projectile from Moving Launch*

In addition to this, the Magnus force has some additional terms. This is all summarized in Table 8.

Component	*Gravity*	*Drag*	*Shear*	*Magnus*
Magnitude	$F_g = m \cdot g$	$F_{drag} = \alpha \cdot v^2$	$F_{shear} = \beta \cdot v$	$F_{magnus} = \gamma \cdot \omega \cdot v$
X component	0	$-\frac{v_x}{v} \cdot F_{drag}$	$-\frac{v_x}{v} \cdot F_{shear}$	$\gamma \cdot (\omega_y \cdot v_z - \omega_z \cdot v_y)$
Y Component	$-m \cdot g$	$-\frac{v_y}{v} \cdot F_{drag}$	$-\frac{v_y}{v} \cdot F_{shear}$	$\gamma \cdot (\omega_z \cdot v_x - \omega_x \cdot v_z)$
Z Component	0	$-\frac{v_z}{v} \cdot F_{drag}$	$-\frac{v_z}{v} \cdot F_{shear}$	$\gamma \cdot (\omega_x \cdot v_y - \omega_y \cdot v_x)$

Table 8 *Forces for Air Resistance on Projectile from Moving Launch*

Finally, the position and velocity can be calculated as follows:

$$v_x(t + \Delta t) = v_x(t) + \frac{F_x(t)}{m} \cdot \Delta t \quad (52)$$

$$v_y(t + \Delta t) = v_y(t) + \frac{F_y(t)}{m} \cdot \Delta t \quad (53)$$

$$v_z(t + \Delta t) = v_z(t) + \frac{F_z(t)}{m} \cdot \Delta t \quad (54)$$

Once the new velocity is known, then the new position can be calculated as well.

$$x(t+\Delta t) = x(t) + \frac{v_x(t) + v_x(t+\Delta t)}{2} \cdot \Delta t \tag{55}$$

$$y(t+\Delta t) = y(t) + \frac{v_y(t) + v_y(t+\Delta t)}{2} \cdot \Delta t \tag{56}$$

$$z(t+\Delta t) = z(t) + \frac{v_z(t) + v_z(t+\Delta t)}{2} \cdot \Delta t \tag{57}$$

The spin is also easy to calculate.

$$\omega_x(t+\Delta t) = \omega_{0x} \cdot e^{-(t+\Delta t)/\tau} \tag{58}$$

$$\omega_y(t+\Delta t) = \omega_{0y} \cdot e^{-(t+\Delta t)/\tau} \tag{59}$$

$$\omega_z(t+\Delta t) = \omega_{0z} \cdot e^{-(t+\Delta t)/\tau} \tag{60}$$

(58) – (60) simplify the spin equations slightly. Rotational torque could technically cause the spin axis to rotate, but this impact is assumed to be small and the spin is assumed to not change direction. Also realize that because of this, the spin in the y direction is zero.

Applying the Models to Measured Data for First Robot

Collecting the Data

To collect the data, the robot was recorded shooting balls. The 2nd, 3rd, and 8th balls were used as data points.

Figure 3 *Robot Shooting Balls*

The balls were recorded using a freeware program called tracker.

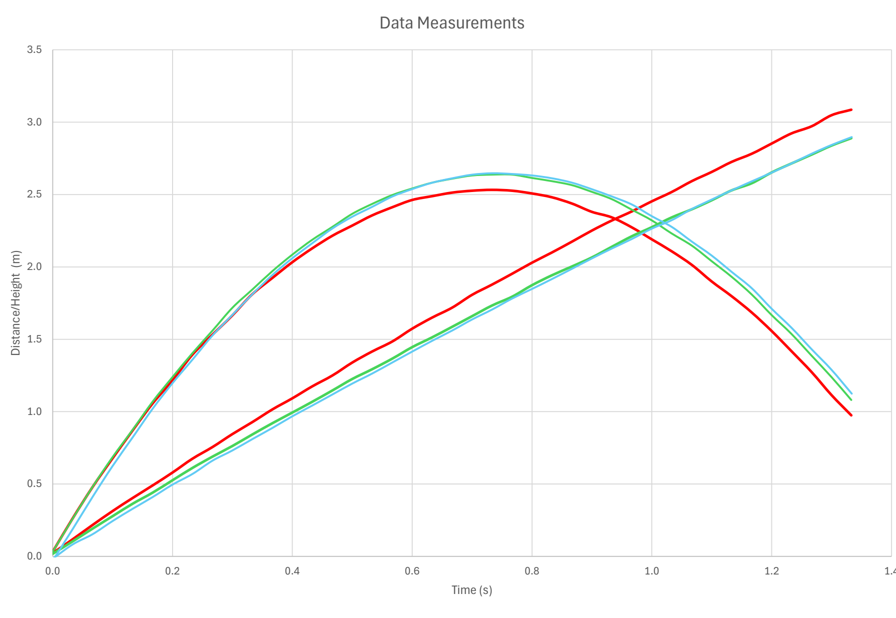


Figure 4 *Measured Ball Trajectories vs. Time*

There is variation between the balls, but Ball 2 seems like an outlier. So, it was decided to take the average of Ball 3 and Ball 8 and declare this to be the measurement. The data for the measurement is shown below.

Time (s)	x (m)	y (m)	Time (s)	x (m)	y (m)	Time (s)	x (m)	y (m)
0.000	0	0	0.500	1.210	2.3565	1.00	2.270	2.337
0.033	0.093	0.224	0.533	1.279	2.4255	1.033	2.333	2.254
0.067	0.173	0.447	0.567	1.353	2.4915	1.067	2.400	2.161
0.100	0.261	0.654	0.600	1.430	2.5395	1.100	2.463	2.060
0.133	0.346	0.848	0.633	1.502	2.582	1.133	2.530	1.949
0.167	0.425	1.045	0.667	1.574	2.611	1.167	2.585	1.831
0.200	0.510	1.217	0.700	1.648	2.6345	1.200	2.652	1.690
0.233	0.590	1.382	0.733	1.720	2.6425	1.233	2.716	1.558
0.267	0.676	1.544	0.767	1.788	2.64	1.267	2.780	1.409
0.300	0.747	1.693	0.800	1.861	2.6235	1.300	2.841	1.265
0.333	0.827	1.828	0.833	1.930	2.6025	1.333	2.894	1.103
0.367	0.904	1.958	0.867	1.996	2.5735			
0.400	0.981	2.072	0.900	2.062	2.528			
0.433	1.056	2.176	0.933	2.134	2.4775			
0.467	1.133	2.271	0.967	2.203	2.413			

Table 9 *Projectile Measured Data*

Attempt to Curve Fit Not Accounting for Air Resistance

For the sake of simplicity and a sanity check, the first check is to try to fit this curve to a model using no air resistance and calculate the angle and speed. The easiest approach is to tweak the initial x and y velocity to get the best fit and then calculate the angle and magnitude of the velocity from this. The parameters that worked best for this are as follows.

Parameter	Value	Unit
v_{0x}	2.35	m/s
v_{0y}	7.2	m/s
g	9.8	m/s^2
v0	7.573804	m/s
θ	71.92389	deg

Table 10 *Model Fitting Parameters for No Air Resistance*

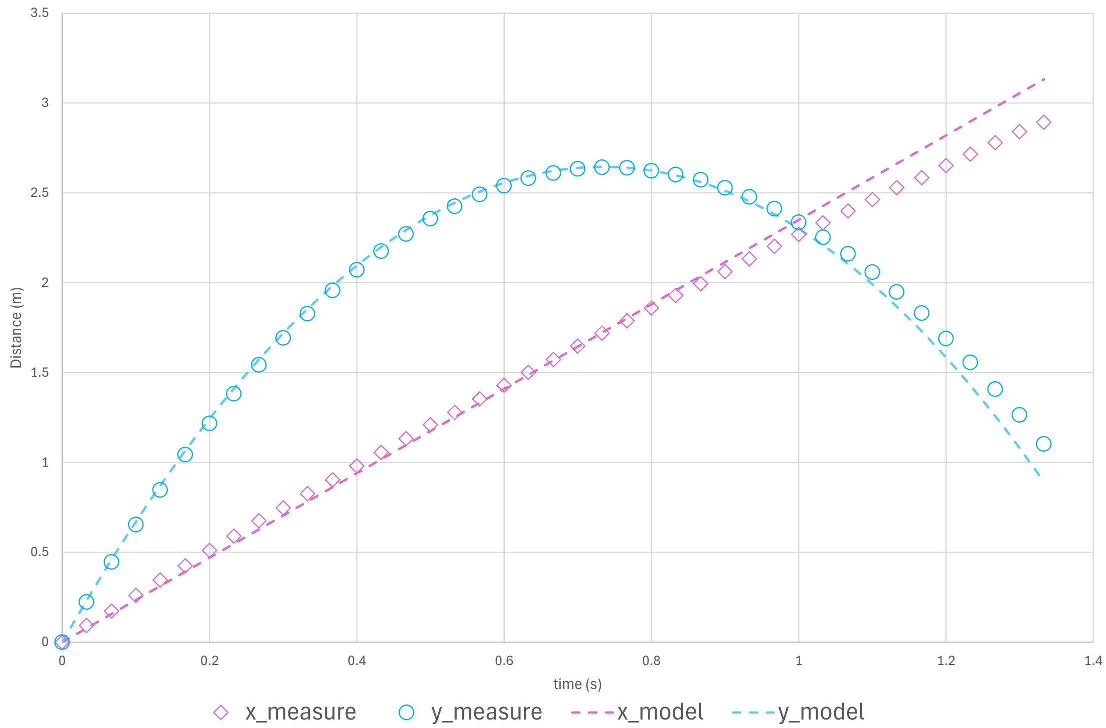


Figure 5 *Matching No Air Resistance X and Y Simulation to Measurement*

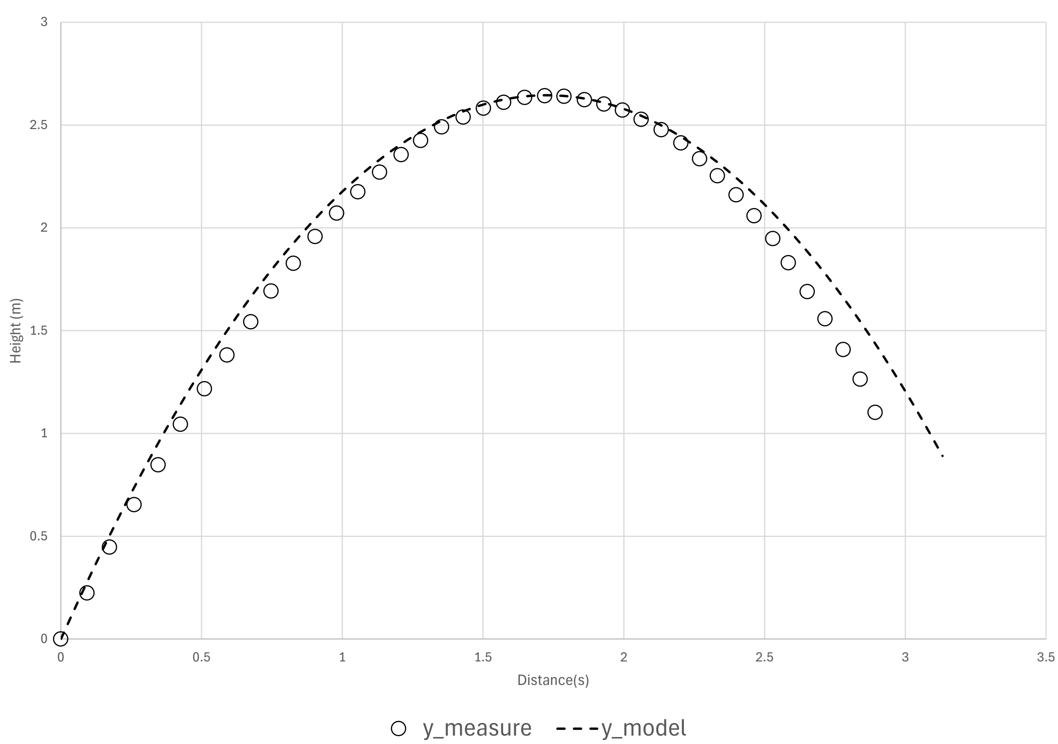


Figure 6 *Matching No Air Resistance Model to Measurement*

Figures 5 and 6 show fair agreement between the model and measured result, but the measurement suggests a slowing in the x and y velocity over time that the no air resistance model does not capture. The ball is rising a little slower than this model predicts and falling a little slower than the model predicts. On the rising part, one could argue that the y velocity is higher than the x and the drag might have something to do with this. On the falling part, the Magnus force would cause the ball to curve inwards. So this is one possible explanation. Figure 7 further suggests that air resistance is relevant as it shows a downward linear trend with the x velocity, which would not be expected if there was no air resistance.

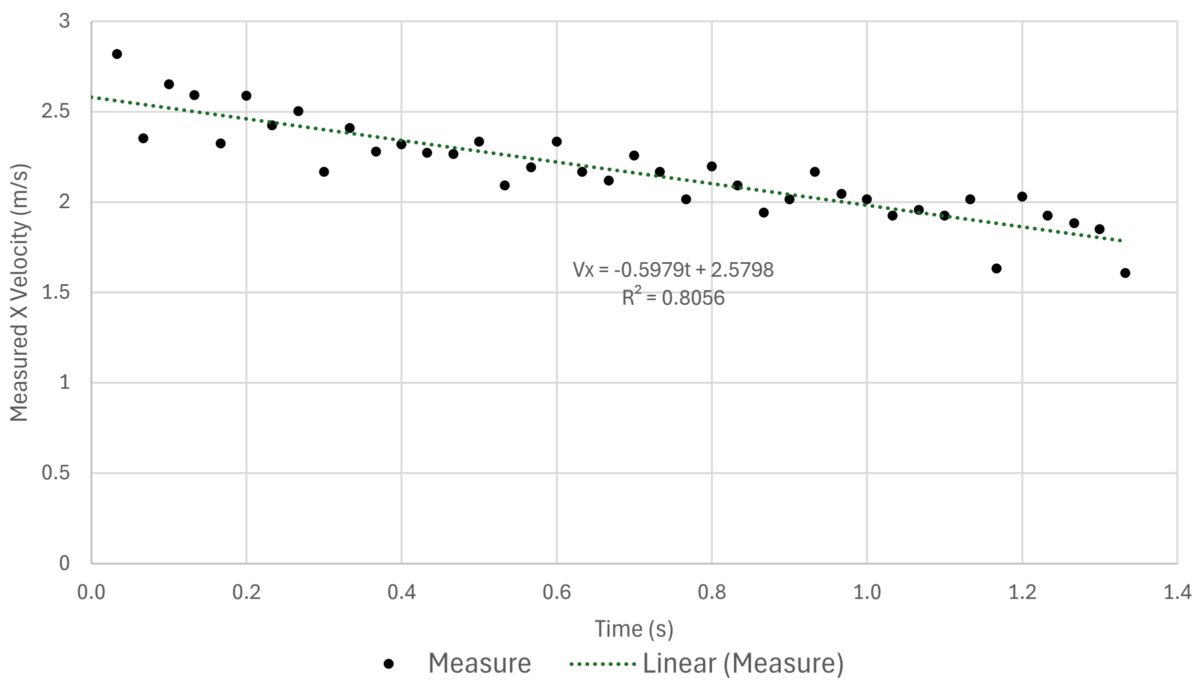


Figure 7 *Downward Trend in X Velocity*

Fitting the Model Accounting for Air Resistance

The best-fit parameters are shown in Table 11, and the resulting agreement to the measurement is shown in Figure 8. This also shows that both the aerodynamic drag force and Magnus force are significant.

Parameter	Symbol	Value	Unit
Initial Velocity	V_0	7.8	m/s
Angle	θ	68.2	deg
Acceleration Due to Gravity	g	9.8	m/s^2
Mass of Ball	m	0.23	kg
Rotational velocity	$\omega / (2\pi)$	15	rev/s
Quadratic Drag Term	α	5.775E-03	$N\cdot(s/m)^2$
Linear Drag Term	β	2.545E-05	N·s/m
Magnus Force Drag Term	γ	6.664E-04	$N\cdot s^2/m$
Spin Decay Coefficient	τ	10	s

Table 11 *Simulation Parameters Accounting for Air Resistance*

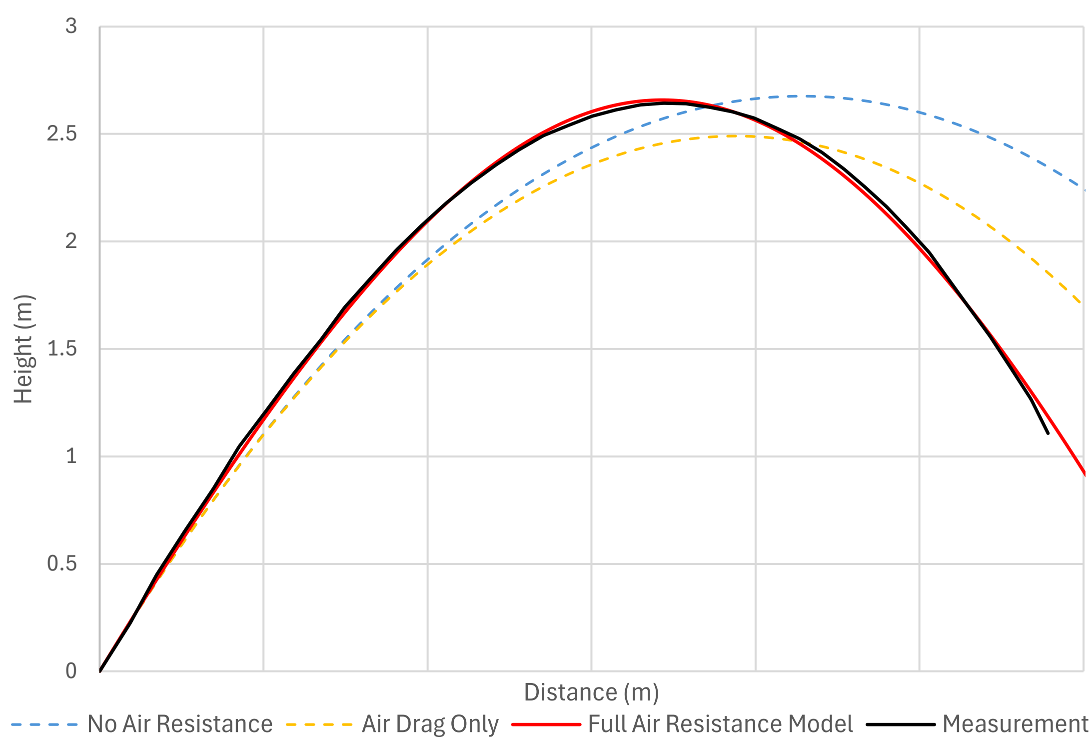


Figure 8 *Matching Simulation to Measured Data*

Calculations for the Second Robot

Overview

These measurements were taken on April 3, 2026 using a robot with a newer shooter design. This second robot was optimized to shoot many more balls, but that came at the cost of the trajectory being much less consistent. Nevertheless, there were some measurements including ball rotation and comparison to what roller speed and hood angle the robot was using.

Calculation of Parameters from Robot Code

For this robot, the actual code was available for the roller speed

$$\frac{\omega}{2\pi}(Revolutions/Second) = 1.5 \cdot 0.8 \cdot d(in\ feet) + 39.705 = 46.33098\ \mathit{Rev/s} \quad (61)$$

Using this value for rotations and the formulae presented in Appendix I, the ball speed and ball rotation can be calculated as shown in Table 12.

Parameter	Formula	Value	Unit
Roller Rotation Speed	$\frac{\omega_1}{2\pi}$	46.331	rev/s
Roller radius	r_1	0.05	m
Ball Radius	r	0.075	m
Roller to Ball Slip factor	k1	1	n/a
Ball to wall Slip factor	k2	1	n/a
Roller Tangential Speed	$v_1 = \omega_1 \cdot r_1$	14.555	m/s
Ball Speed	$v_0 = \frac{k_1}{1+k_2} \cdot v_1$	7.2777	m/s
Ball Rotation	$\omega = \frac{k_1 \cdot k_2}{1+k_2} \cdot \frac{v_1}{r}$	15.444	rev/s

Table 12 *Calculation of Ball Speed and Rotation from Robot Code*

The hood angle, which is measured somewhat from the vertical (not horizontal like θ) for the robot code is calculated as follows:

$$Hood\ Angle\ (deg) = 1.3198 \cdot d(in\ feet) + 4.2132 = 11.501 \quad (62)$$

Collection of Data

Although the ball trajectories were much less consistent, one ball was tracked with the understanding that there is less confidence in this measurement than the first robot. The curve was fit and these were found to be a close match. One observation for Figure 10 is that the air resistance is significant and the spin on the ball contributes a good amount to this.

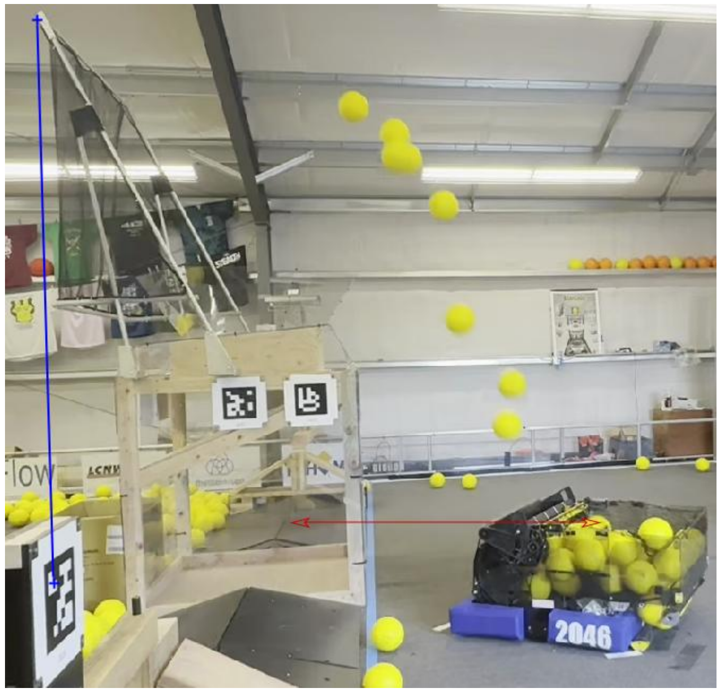


Figure 9 *Second Robot Shooting Balls from a distance of 1.683 m = 5.52165 ft*

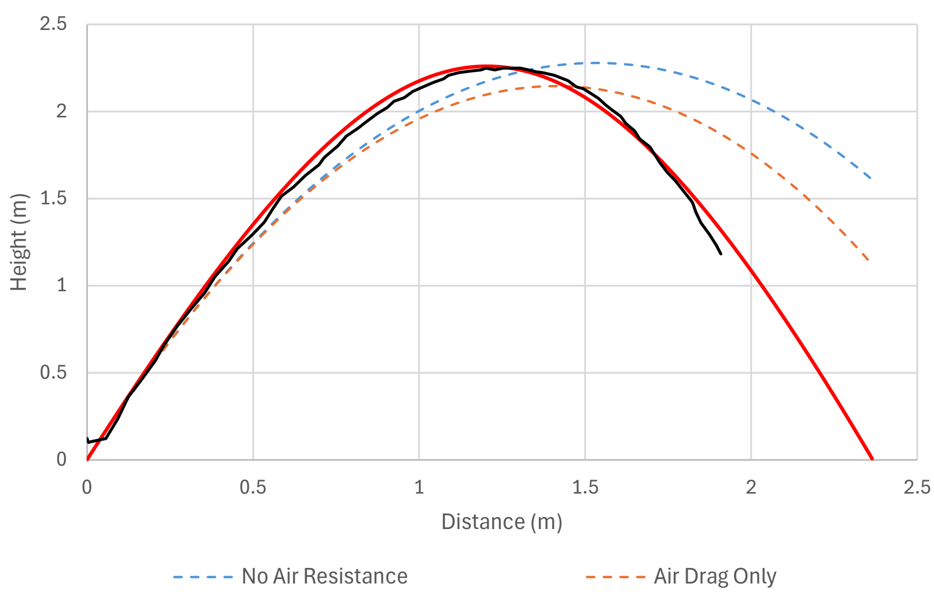
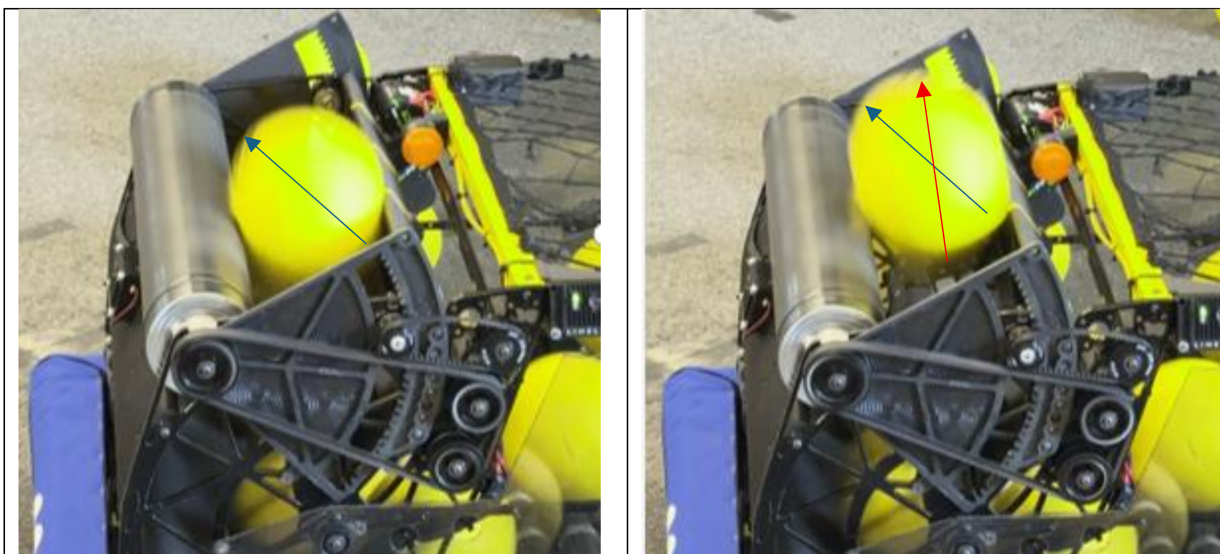


Figure 10 *Matching Simulation to Measured Data*

Parameter	Symbol	Value	Unit
Initial Velocity	v_0	7.05	m/s
Launch Angle	θ	71.4	deg
Acceleration Due to Gravity	g	9.8	m/s^2
Target Distance	d	1.683	m
Target Height	h	1.2	m
Mass of Ball	m	0.23	kg
Rotational velocity	$\omega / (2\pi)$	15.4	rev/s
Spin Decay Coefficient	τ	10	s
Quadratic Drag Term	α	5.775E-03	$N\cdot(s/m)^2$
Linear Drag Term	β	2.545E-05	$N\cdot(s/m)$
Magnus Force Drag Term	γ	6.664E-04	$N\cdot s^2/m$

Table 13 *Estimation of Parameters from Measured Data*

For safety reasons, the robot could not have a video taken of it while shooting distance for the other pictures. However, it was measured at a slightly different distance of about 4 ft for a sanity check. At this distance, using (61) and the supporting equations, this works out to a spin of 14.835 rev/s. Figure 11 shows a measurement of about 45 degrees rotation in 0.01 second that works out to about 12 rev/s. So this calculation seems roughly correct.



T = 0.01 s | T = 0.02 s

Figure 11 *Measurement of Ball Rotation*

Summary of Finding from Second Robot

Parameter	Symbol	Robot Code	Measured	Unit
Initial Velocity	v_0	7.278	7.05	m/s
Launch Angle	θ	-	71.4	deg
Hood Angle	θ_{hood}	11.5	-	deg
Acceleration Due to Gravity	g		9.8	m/s^2
Target Distance	d	1.683	1.683	m
Target Height	h		1.2	m
Mass of Ball	m		0.23	kg
Rotational velocity	$\omega / (2\pi)$	15.444	15.4	rev/s
Spin Decay Coefficient	τ		10	s
Quadratic Drag Term	α		5.78E-03	$N\cdot(s/m)^2$
Linear Drag Term	β		2.54E-05	$N\cdot(s/m)$
Magnus Force Drag Term	γ		6.664E-04	$N\cdot(s^2/m)$

Table 14 *Comparison of Robot Code to Measured Data*

So from Table 14, the initial velocity and rotational velocity seem to match fairly well. As the balls were being slightly compressed, this might be an explanation why the initial measured velocity was slightly lower than the value in the code and the measured rotational velocity was slightly higher.

The robot code uses the 'Hood Angle', which is based on the angle of the hood that the ball rolls against, and this angle is referenced to the vertical. So as the hood angle decreases, the launch angle increases. This is measured more from the vertical than the horizontal. Based on the results in Table 14, these two angles can be related with (63).

$$\theta \cong 90 - 7.1 - \theta_{hood} \tag{63}$$

Realize that these findings are specific to this second robot and specific to the particular version of the code that was uploaded to the robot of the time of measurement.

Conclusion

The trajectory model developed in this document provides a practical balance between analytical simplicity and physical realism.

The no-air-resistance model offers useful intuition and closed-form solutions, but comparison with measured data shows that aerodynamic effects—particularly drag—play a significant role in the observed trajectory. Incorporating these effects improves agreement with experimental results and better captures the reduction in horizontal velocity.

The model also highlights the importance of understanding how initial conditions, including launch angle, velocity, and spin, influence system performance. The Magnus force is modeled using the cross product of spin and velocity, allowing its direction to evolve naturally with the trajectory.

Several simplifying assumptions were used in the analysis. First, the spin axis of the ball is assumed to remain fixed in direction during flight. While aerodynamic torque can, in principle, alter the spin axis, this effect is neglected because the flight time is short relative to the estimated spin decay time constant. Second, the aerodynamic coefficients used in the drag, shear, and Magnus force models are assumed constant. In reality, these coefficients depend on factors such as air density, temperature, humidity, and Reynolds number. However, for the range of conditions considered, these variations are relatively small and are not expected to significantly impact the results.

Overall, the model is sufficient for understanding system behavior and improving performance, and it provides a foundation for future refinements such as modeling spin decay and extending the numerical solution to fully account for motion during launch.

The author would like to thank the Tahoma robotics team for the use of their robot and for providing a practical setting in which to explore these modeling ideas.

Appendix A Example for Stationary Turret with NO Air Resistance

Parameter	**Symbol/Formula**	**Value**		**Unit**
Initial Velocity	v_0	7.5		m/s
Gravity Constant	g	9.8		m/s^2
Target Distance	d	3		m
Target Height	h	1.2		m
Intermediate Constants	$a = g \cdot d^2$	88.2		m^3/s^2
	$b = -2 \cdot {v_0}^2 \cdot d$	-337.5		m^3/s^2
	$c = 2 \cdot {v_0}^2 \cdot h + g \cdot d^2$	223.2		m^3/s^2
Angle	$\theta_{1,2} = tan^{-1}\left(\frac{-b \pm \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a}\right)$	71.4281	40.3733	deg
Peak Time	$t_{Peak} = \frac{v_0 \cdot sin\theta}{g}$	0.7255	0.4957	s
Peak Height	$h_{Peak} = t_{Peak} \cdot v_0 \cdot sin\theta - \frac{1}{2} g \cdot {t_{Peak}}^2$	2.5788	1.2042	m
Time to Target	$t_{Target} = \frac{d}{v_0 \cdot cos\theta}$	1.2559	0.5250	s
Time to Hit Ground	$0 = t_{Ground} \cdot v_0 \cdot sin\theta - \frac{1}{2} g \cdot {t_{Ground}}^2$	1.4509	1.4509	s
Range	$Range = v_0 \cdot sin\theta \cdot t_{Ground}$	3.4658	3.4658	m
Target Strike Slope	$m_{1,2} = \frac{dy}{dx} = tan\theta - \frac{g \cdot d}{{v_0}^2 \cdot cos^2\theta}$	-2.1763	-0.0503	n/a
Target Strike Angle	$\phi_{1,2} = tan^{-1}(m_{1,2})$	-124.6909	-2.8799	deg

Appendix B Solution Method for the Quartic Equation

The general quartic equation is of the form:

$$x^4 + A_3 \cdot x^3 + A_2 \cdot x^2 + A_1 \cdot x + A_0 = 0 \qquad \textit{(B1)}$$

The following substitution can be made to eliminate the cubic term.

$$t - \frac{A_3}{4} \rightarrow x \qquad \textit{(B2)}$$

(B3) shows the result, which is called the depressed quartic equation. The solution of the depressed quartic equation is credited to the mathematician Lodovico Ferrari.

$$t^4 + A \cdot t^2 + B \cdot t + C = 0 \qquad \textit{(B3)}$$

The trick to solving this is to write it in the form:

$$(t^2 + u_0)^2 = (2 \cdot u_0 - A) \cdot t^2 - B \cdot t + {u_0}^2 - C \qquad \textit{(B4)}$$

Now u_0 is strategically chosen to make the right hand side a perfect square. However, to find the value that works, it requires the solution of a cubic equation.

$$u^3 + a_2 \cdot u^2 + a_1 \cdot u + a_0 = 0 \qquad \textit{(B5)}$$

$$a_2 = -\frac{A}{2} \qquad \textit{(B6)}$$

$$a_1 = -B \qquad \textit{(B7)}$$

$$a_0 = -\frac{4 \cdot A \cdot C - B^2}{8} \qquad \textit{(B8)}$$

The solution method of the cubic polynomial is credited to the mathematician Gerolamo Cardano. The general approach is to eliminate the a2 term with the substitution:

$$y - \frac{a_2}{3} \rightarrow u \qquad \textit{(B9)}$$

And then use a substitution introduced by the mathematician Viete

$$w - \frac{1}{w} \rightarrow y \qquad \textit{(B10)}$$

This leads to a quadric equation that can be solved. The solution method is simplified by introducing some intermediate variables.

$$Q = \frac{3 \cdot a_1 - {a_2}^2}{9} \tag{B11}$$

$$R = \frac{9 \cdot a_1 \cdot a_2 - 2 \cdot {a_2}^3 - 27 \cdot a_0}{54} \tag{B12}$$

$$D = R^2 + Q^3 \tag{B13}$$

Now the solution approach differs depending on the value of D. If D>=0, then there is one real root to the cubic (B16) and it can be found as follows:

$$S = \sqrt[3]{R + \sqrt{D}} \tag{B14}$$

$$T = \sqrt[3]{R - \sqrt{D}} \tag{B15}$$

$$u_0 = S + T - \frac{a_2}{3} \tag{B16}$$

$$u_1 = -\frac{S+T}{2} - \frac{a_2}{3} - i \cdot \frac{\sqrt{3} \cdot (S-T)}{2} \tag{B17}$$

$$u_2 = -\frac{S+T}{2} - \frac{a_2}{3} + i \cdot \frac{\sqrt{3} \cdot (S-T)}{2} \tag{B18}$$

The other roots as found in (B17) and (B18) are complex and are not necessary to solve the 4th order polynomial, but are included for academic interest only.

In the case that D<0, (B14) – (B18) involve complex variables and it can be confusing to figure out which root to take (although it is possible). In this case, it is easier to express this using a trigonometric approach as shown in (B17) – (B22).

$$\varphi = cos^{-1}\left(\frac{R}{-\sqrt{Q^3}}\right) \tag{B19}$$

$$u_0 = 2 \cdot \sqrt{-Q} \cdot cos\left(\frac{\varphi}{3}\right) - \frac{a_2}{3} \tag{B20}$$

$$u_0 = 2 \cdot \sqrt{-Q} \cdot cos\left(\frac{\varphi + 2\pi}{3}\right) - \frac{a_2}{3} \qquad \text{(B21)}$$

$$u_0 = 2 \cdot \sqrt{-Q} \cdot cos\left(\frac{\varphi + 4\pi}{3}\right) - \frac{a_2}{3} \qquad \text{(B22)}$$

In the case that D ≤ 0 and there are multiple roots, choose one that makes expression (B23) real. Once a cubic root is found and decided on, calculate the next intermediate variable.

$$p = \sqrt{2 \cdot u_0 - A} \qquad \text{(B23)}$$

This leads to two quadratic equations.

$$t^2 \pm p \cdot t + u_0 \mp \frac{B}{2 \cdot p} = 0 \qquad \text{(B24)}$$

The solution of this leads to 4 roots, although some of them may be negative or complex.

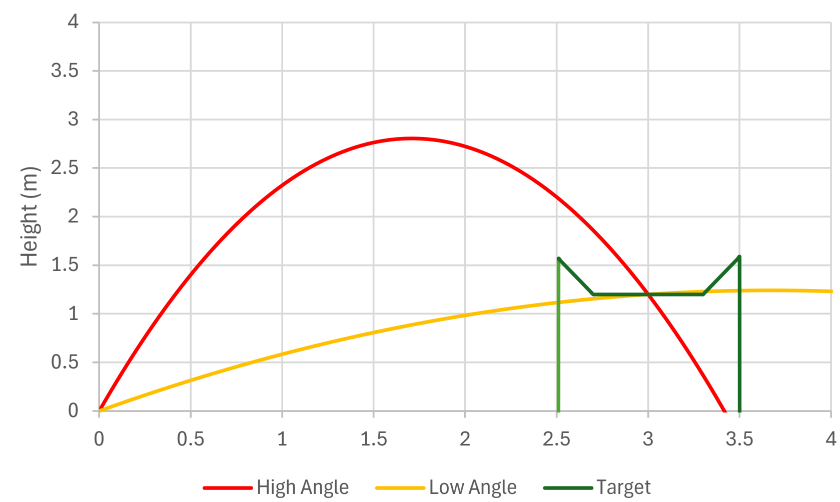
The quartic equation was first solved by mathematician Lodovico Ferrari (1522–1565), which reduces the quartic to a resolvent cubic equation. The cubic is then solved using Cardano's formula, first published by Gerolamo Cardano (1501–1576) in *Ars Magna* (1545), where Ferrari's quartic solution also appeared. Niels Henrik Abel (1802–1829), proved that no general closed-form algebraic solution exists for polynomial equations of degree five or higher.

Appendix C Example for Moving Turret no Air Resistance

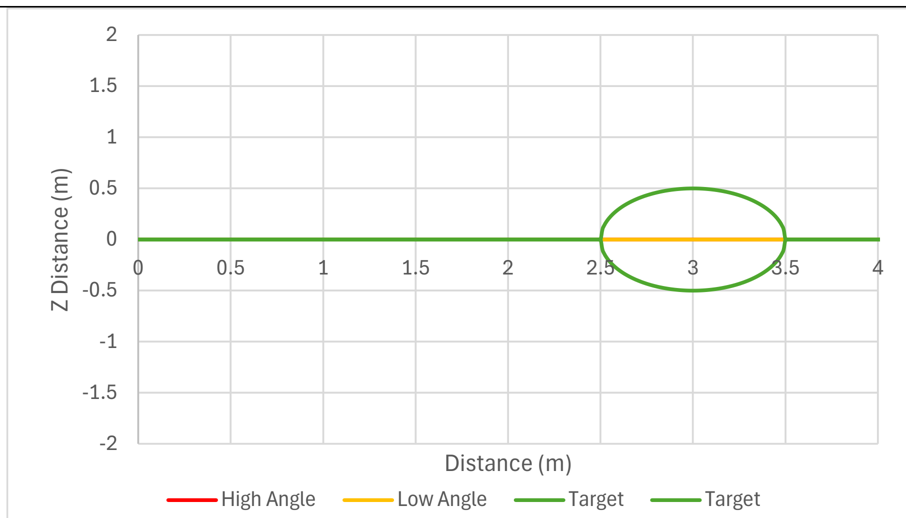
<table>
<tr><th>Parameter</th><th>Symbol/Formula</th><th colspan="2">Value</th><th>Unit</th></tr>
<tr><td>Initial Velocity</td><td>v_0</td><td colspan="2">7.5</td><td>m/s</td></tr>
<tr><td>Gravity Constant</td><td>g</td><td colspan="2">9.8</td><td>m/s^2</td></tr>
<tr><td>Target Distance</td><td>d</td><td colspan="2">3</td><td>m</td></tr>
<tr><td>Target Height</td><td>h</td><td colspan="2">1.2</td><td>m</td></tr>
<tr><td>Vehicle Speed</td><td>v_m</td><td colspan="2">2</td><td>m/s</td></tr>
<tr><td>Vehicle heading angle</td><td>ϕ_m</td><td colspan="2">30</td><td>deg</td></tr>
<tr><td rowspan="3">Quartic Equation</td><td>$A = \frac{4 \cdot (g \cdot h + {v_m}^2 - {v_0}^2)}{g^2}$</td><td colspan="2">-1.68638</td><td rowspan="16">Units Consistent with Final Equation</td></tr>
<tr><td>$B = -\frac{8 \cdot d \cdot v_m \cdot cos\phi_m}{g^2}$</td><td colspan="2">-0.43283</td></tr>
<tr><td>$C = \frac{4 \cdot (d^2 + h^2)}{g^2}$</td><td colspan="2">0.43482</td></tr>
<tr><td rowspan="10">Cubic Equation</td><td>$a_2 = -\frac{A}{2}$</td><td colspan="2">0.84319</td></tr>
<tr><td>$a_1 = -B$</td><td colspan="2">-0.43482</td></tr>
<tr><td>$a_0 = -\frac{4 \cdot A \cdot C - B^2}{8}$</td><td colspan="2">-0.39005</td></tr>
<tr><td>$Q = \frac{3 \cdot a_1 - {a_2}^2}{9}$</td><td colspan="2">-0.2239366</td></tr>
<tr><td>$R = \frac{9 \cdot a_1 \cdot a_2 - 2 \cdot {a_2}^3 - 27 \cdot a_0}{54}$</td><td colspan="2">0.1117178</td></tr>
<tr><td>$D = R^2 + Q^3$</td><td colspan="2">0.001251</td></tr>
<tr><td>$S = \sqrt[3]{R + \sqrt{D}}$</td><td colspan="2">0.52787</td></tr>
<tr><td>$T = \sqrt[3]{R - \sqrt{D}}$</td><td colspan="2">0.42423</td></tr>
<tr><td>$u_0 = S + T - \frac{a_2}{3}$</td><td colspan="2">0.67103</td></tr>
<tr><td rowspan="3">Quadratic Equation</td><td>a</td><td>1</td><td>1</td></tr>
<tr><td>b</td><td>1.74024</td><td>-1.74024</td></tr>
<tr><td>c</td><td>0.79539</td><td>0.54667</td></tr>
</table>

Parameter	**Symbol/Formula**	**Value1**	**Value2**	**Unit**
t	Use quadratic formula and take real roots	1.32886	0.41139	s
Turret Velocities	$v_{0x} = \frac{d}{t} - v_m \cdot cos\phi_m$	0.52553	5.56036	m/s
	$v_{0y} = \frac{h}{t} + \frac{1}{2} \cdot g \cdot t$	7.41443	4.93276	m/s
	$v_{0z} = -v_m \cdot sin\phi_m$	-1.00000	-1.00000	m/s
Elevation Angle	$\theta = tan^{-1}\left(\frac{v_y}{\sqrt{{v_{0x}}^2 + {v_{0z}}^2}}\right)$	81.33689	41.12492	deg
Azimuth Angle	$\phi = tan^{-1}\left(\frac{v_{0z}}{v_{0x}}\right)$	-62.27674	-10.19535	deg

XY Plane Graph



XZ Plane Graph



Appendix D Derivation of Drag Force for Air Resistance

This appendix explains why the air drag force is proportional to the square of velocity. It uses two approaches, the first relating momentum to impulse and the second relating work to kinetic energy.

Variables:

Symbol	***Parameter***	***Unit***	***Typical Value***
Δt	Time interval	s	n/a
m	Mass	kg	n/a
ρ	Air Density	kg/m^3	1.257
v	Velocity	m/s	n/a
F	Force	N	n/a
A	Cross sectional area	m^2	n/a
C_d	Drag Coefficient	n/a	0.52 (ball)

Momentum – Impulse Approach

$$Momentum = Mass \times Velocity \quad \text{(D1)}$$

The velocity is a simple constant. Now consider the mass of air over a period of time, Δt. At time zero, no mass has acted. At the end, it would be the time period x density x velocity. So the average would be the ½ of the max. In other words

$$Momentum = \frac{1}{2} \cdot m \cdot v = \frac{1}{2} \cdot (\rho \cdot A \cdot v \cdot \Delta t) \cdot v \quad \text{(D2)}$$

Another quantity is impulse, which is numerically the same as momentum. In other words:

$$Momentum = Impulse = F \cdot \Delta t \quad \text{(D3)}$$

Equating (D2) to (D3) we get:

$$F \cdot \Delta t = \frac{1}{2} \cdot (\rho \cdot A \cdot v \cdot \Delta t) \cdot v \quad \text{(D4)}$$

$$F = \frac{1}{2} \cdot \rho \cdot C_d \cdot A \cdot v^2 \quad \text{(D5)}$$

C_d is a constant that is introduced to account for the shape of the object. For a ball, C_d = 0.52.

Work – Kinetic Energy Approach

$$Energy = Work = Kinetic\ Energy \tag{D6}$$

Recall that:

$$Kinetic\ Energy = \frac{1}{2} \cdot m \cdot v^2 = \frac{1}{2} \cdot (\rho \cdot A \cdot v \cdot \Delta t) \cdot v^2 \tag{D7}$$

Equating (D6) and (D7) give the relationship:

$$Work = Force\ \times Distance = \ F \cdot (v \cdot \Delta t) \tag{D8}$$

Equating (D7) to (D6) we get:

$$F \cdot (v \cdot \Delta t) = \frac{1}{2} \cdot (\rho \cdot A \cdot v \cdot \Delta t) \cdot v^2 \tag{D9}$$

Simplifying (D9) and introducing the drag coefficient C_d, to account for the shape of the surface and the roughness of the surface yields:

$$F = \frac{1}{2} \cdot \rho \cdot C_d \cdot A \cdot v^2 \tag{D10}$$

Final Formula

At the end of the day, it is easier to compress this into a single formula

$$F = \alpha \cdot v^2 \tag{D11}$$

$$\alpha = \frac{1}{2} \cdot \rho \cdot C_d \cdot A \tag{D12}$$

Determining the Direction of the Drag Force

One key point is to understand that because the drag force is not linear with velocity, one cannot calculate the drag force independently for x and y directions based on the x and y velocities. Instead, one must first calculate the total magnitude and divide it down.

$$F_x = -\frac{v_x}{|v|} \cdot F \tag{D13}$$

$$F_y = -\frac{v_y}{|v|} \cdot F \tag{D14}$$

Appendix E Derivation of Shear Air Force

Symbol	***Parameter***	***Unit***	***Typical Value***
ρ	Air density	kg/m^3	1.257
μ	Air viscosity	Pa·s	1.8×10^{-5}
v	Velocity	m/s	n/a
η	Coupling Coefficient	n/a	0.4
F	Force	N	n/a

Consider the air flow along a long surface and think of the air being in different layers around the object, known as laminar flow. For the layer closest to the surface, the coupling coefficient, η, determines how close this layer matches the speed of the object. For farther layers, they exert some force and this is influenced by the viscosity of the air.

$$Force = Viscosity \times (Coupling\ Coefficient) \times (Area\ Exposed\ to\ Fluid) \times Velocity \qquad (E1)$$

For instance, for a "long" cylinder, this works out to

$$F_{Cylinder} = \mu \cdot \eta \cdot 2\pi \cdot r \cdot L \cdot v \qquad (E2)$$

For a sphere, the velocity of the fluid is not the same at all points. The derivation involves the Navier-Stokes equation and is beyond the scope of this document. It already accounts for an intermediate range for the coupling coefficient, so this is built in the formula.

$$F_{Sphere} = \mu \cdot 6\pi \cdot r \cdot v \qquad (E3)$$

The Reynolds number is an indication of if we can assume all these equations. If it is much greater than one, then it turns out that these effects are negligible relative to the drag force, but we can still use a linear term as an empirical correction term.

$$Reynolds\ Number = Re = \frac{Fluid\ Inertia}{Viscosity} = \frac{\rho \cdot v \cdot L}{\mu} \qquad (E4)$$

So for a sphere, this is easily calculated

$$Re_{Sphere} = \frac{2 \cdot r \cdot \rho \cdot v}{\mu} \qquad (E5)$$

At the end of the day, the shear force can be calculated as:

$$F = \beta \cdot v \qquad (E6)$$

For the case of a sphere and Reynolds number less than one we can say that:

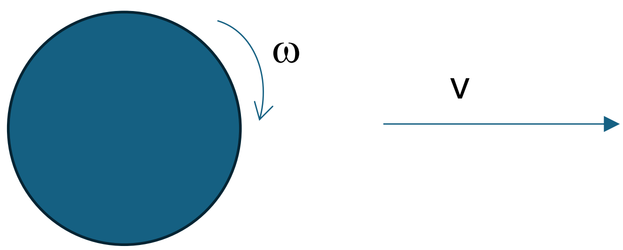
$$\beta = 6\pi \cdot \mu \cdot r \qquad (E7)$$

Appendix F Derivation of Magnus Force for Air Resistance

The Magnus force acts on spinning objects and causes them to curve in the air because the air pressure is different on opposite sides of the object.

Symbol	***Parameter***	***Unit***	***Typical Value***
Δt	Time interval	s	n/a
m	Mass	kg	n/a
ρ	Air Density	kg/m^3	1.257
v	Velocity	m/s	n/a
η	Air Coupling Coefficient	n/a	0.4
F	Magnus Force	N	n/a

Deriving Magnitude of Magnus Force for a Sphere



The first step is understanding Bernoulli's equation, which relates pressure to the square of velocity.

$$P = \frac{1}{2}\rho \cdot {v_{Effective}}^2 \qquad \textit{(F1)}$$

The next step is to calculate the effective velocities at the top and bottom of the object.

$$v_{Top} = v + \eta \cdot \omega \cdot r \qquad \textit{(F2)}$$
$$v_{Bottom} = v - \eta \cdot \omega \cdot r \qquad \textit{(F3)}$$

The difference in pressure can be calculated as follows:

$$P = \frac{1}{2} \cdot \rho \cdot {v_{Top}}^2 - \frac{1}{2} \cdot \rho \cdot {v_{Bottom}}^2 \qquad \textit{(F4)}$$
$$P = \frac{1}{2} \cdot \rho \cdot [(v + \eta \cdot \omega \cdot r)^2 - (v - \eta \cdot \omega \cdot r)^2] = 2 \cdot \rho \cdot v \cdot \eta \cdot \omega \cdot r_{Effective} \qquad \textit{(F5)}$$

So now expand (F5) for an arbitrary point on the sphere, not just the top or bottom point. The effective radius is that about the axis of rotation.

$$r_{Effective} = r \cdot sin\theta \cdot sin\phi \qquad \textit{(F6)}$$

Furthermore, all the components cancel out except in the force in the up/down direction, so we need to know that projection.

$$Projection = sin\theta \cdot sin\phi \tag{F7}$$

Combining (F5), (F6), and (F7) yield the following result:

$$P_{Effective} = \ 2 \cdot \rho \cdot v \cdot \eta \cdot \omega \cdot r \cdot sin^2\theta \cdot sin^2\phi \tag{F8}$$

Now recall the general definition of pressure.

$$Pressure = \frac{Force}{Area} \tag{F9}$$

If the pressure was constant over the whole area, one could just solve (F9) for the force. However, since this is not the case, the approach is to consider it constant over a small area, *dA*. This surface area created by sweeping the angles of θ and ϕ is obtained by multiplying the two arc lengths.

$$dA = (r \cdot d\theta) \times (r \cdot sin\theta \cdot d\phi) = r^2 \cdot sin\theta \cdot d\theta \cdot d\phi \tag{F10}$$

So the force acting on this infinitesimal area can be found by combining (F8), (F9), and (F10).

$$dF = P_{Effective} \cdot dA = 2 \cdot \rho \cdot v \cdot \eta \cdot \omega \cdot r^3 \cdot sin^3\theta \cdot sin^2\phi \cdot d\theta \cdot d\phi \tag{F11}$$

So now the next step is to integrate this over the upper half of the sphere.

$$F = \int P \cdot dA = 2 \cdot \rho \cdot v \cdot \eta \cdot \omega \cdot r^3 \cdot \int_0^\pi \int_0^\pi sin^3\theta \cdot sin^2\phi \cdot d\theta \cdot d\phi$$

$$= 2 \cdot \rho \cdot v \cdot \eta \cdot \omega \cdot r^3 \cdot \int_0^\pi \int_0^\pi (1 - cos^2\theta) \cdot sin\theta \cdot \left(\frac{1 - cos2\phi}{2}\right) \cdot d\theta \cdot d\phi$$

$$= 2 \cdot \rho \cdot v \cdot \eta \cdot \omega \cdot r^3 \cdot \left(-cos\theta + \frac{cos^3\theta}{3}\right)\Bigg|_0^\pi \cdot \left(\frac{\phi}{2} - \frac{sin2\phi}{4}\right)\Bigg|_0^\pi \tag{F12}$$

$$= 2 \cdot \rho \cdot v \cdot \eta \cdot \omega \cdot r^3 \cdot \left(2 - \frac{2}{3}\right)\Bigg| \cdot \left(\frac{\pi}{2}\right)$$

$$= \frac{4\pi}{3} \cdot \rho \cdot v \cdot \eta \cdot \omega \cdot r^3$$

However, the result in (F12) is based on an idealized pressure distribution over the entire sphere. In practice, the flow over a sphere is three-dimensional. While the front portion of the sphere experiences the strongest interaction with the incoming flow, the aft side is affected by flow separation, wake formation, and air moving around the sides of the sphere. These effects reduce the pressure difference between the top and bottom surfaces on the rear portion of the ball compared to the front.

As a result, the effective contribution of the rear portion of the sphere to the net Magnus force is reduced relative to the idealized model. Therefore, the actual force is expected to lie between approximately one-half of the full integrated value in (F12) and the full value itself.

$$\frac{2\pi}{3} \cdot \rho \cdot v \cdot \eta \cdot \omega \cdot r^3 \leq F \leq \frac{4\pi}{3} \cdot \rho \cdot v \cdot \eta \cdot \omega \cdot r^3 \quad \textit{(F13)}$$

So from a practical standpoint, one can just choose the middle value to get a nice numerical result.

$$F = \rho \cdot v \cdot \eta \cdot \omega \cdot \pi \cdot r^3 \quad \textit{(F14)}$$

This can be rewritten as follows:

$$F = \gamma \cdot v \cdot \omega \quad \textit{(F15)}$$

$$\gamma = \rho \cdot \eta \cdot \pi \cdot r^3 \quad \textit{(F16)}$$

Determining Direction of Magnus Force

The Magnus force always points perpendicular to the velocity and perpendicular to the spin. Note that the spin is represented as a vector using the right-hand rule. A compact way to say this is:

$$\vec{F} = \gamma \cdot \vec{\omega} \times \vec{v} \quad \textit{(F17)}$$

This can be expressed in terms of components as well.

$$\vec{F} = \gamma \cdot \vec{\omega} \times \vec{v} = \gamma \cdot \begin{vmatrix} i & j & k \\ \omega_x & \omega_y & \omega_z \\ v_x & v_y & v_z \end{vmatrix} = \gamma \cdot \left(\omega_y \cdot v_z - \omega_z \cdot v_y, \omega_z \cdot v_x - \omega_x \cdot v_z , \omega_x \cdot v_y - \omega_y \cdot v_x \right) \quad \textit{(F18)}$$

For example, consider the case where there is no motion in the z direction and the spin is pure backspin.

$$\vec{F} = \gamma \cdot \vec{\omega} \times \vec{v} = \gamma \cdot \begin{vmatrix} i & j & k \\ 0 & 0 & \omega \\ v_x & v_y & 0 \end{vmatrix} = \left(-\gamma \cdot v_y \cdot \omega , \gamma \cdot v_x \cdot \omega , 0 \right) \quad \textit{(F19)}$$

If the result of (F19) is applied to the trajectory, the Magnus force reduces the projectile's x-velocity during ascent while exerting a positive force in the y direction. During descent, it increases the x-velocity while continuing to provide an upward force in the y direction.

Accounting for Spin Decay

The Magnus force depends on the spin of the projectile. However, there is an aerodynamic torque that will cause this spin to slow down. This torque is assumed to be proportional to the angular velocity. The value of k is difficult to calculate directly and depends on many factors but is easy to extract from measured data.

$$F_{Torque} = -k \cdot \omega \tag{F20}$$

The rotational equivalent of Newton's second law is:

$$F_{Torque} = I \cdot \frac{d\omega}{dt} \tag{F21}$$

Combining (F20) and (F21) yields a differential equation.

$$\tau \cdot \frac{d\omega}{dt} + \omega = 0 \tag{F22}$$

$$\tau = \frac{I}{k} \tag{F23}$$

(F22) can be solved using the separation of variables technique

$$\frac{d\omega}{\omega} = -\frac{1}{\tau} \cdot dt \tag{F24}$$

$$Ln(\omega) = -\frac{t}{\tau} + c \tag{F25}$$

The final solution is as follows.

$$\omega = \omega_0 \cdot e^{-t/\tau} \tag{F26}$$

The parameter τ can be estimated from measured data. From an intuitive standpoint, this is the time at which the spin has slowed to about 37% of its initial value. If the spin is acting in more than one direction, (F26) can be applied independently to each component. In the case of motion with a nonzero z component, aerodynamic torque can in principle alter the direction of the spin axis. However, if the flight time is much shorter than the time constant, τ, this effect can be assumed to be negligible.

Appendix G Calculation of Reynolds Number, α, β, and γ

For the examples used throughout this paper, this appendix shows calculations for the Reynolds number, α, β, and γ. A representative velocity is used to evaluate the Reynolds number. Since the Reynolds number is much greater than one, the exact value of velocity has minimal impact on this conclusion.

Symbol	***Parameter***	***Formula***	***Typical Value***	***Unit***
v_0	Initial velocity	n/a	7.5	m/s
r	Radius of ball		0.075	m
ρ	Air density		1.257	kg/m^3
μ	Air viscosity		1.8×10^{-5}	$Pa \cdot s$
η	Air coupling constant		0.4	n/a
C_d	Ball drag coefficient		0.52	n/a
A	Cross sectional area	$A = \pi r^2$	0.01767	m^2
L	Length	$L = 2 \cdot r$	0.15	m
Re	Reynolds number	$Re = \frac{\rho \cdot v_0 \cdot L}{\mu}$	78600	n/a
α	Drag coefficient	$\frac{1}{2} \cdot \rho \cdot C_d \cdot A$	5.775×10^{-3}	$N \cdot \left(\frac{s}{m}\right)^2$
β	Shear coefficient	$\beta = 6\pi \cdot \mu \cdot r$	2.545×10^{-5}	$N \cdot \frac{s}{m}$
γ	Magnus coefficient	$\gamma = \pi \cdot \rho \cdot \eta \cdot r^3$	6.664×10^{-4}	$N \cdot (s^2/m)$

Appendix H Accounting for Surface Roughness for Air Resistance

One may ask questions such as: if all rivets and seams were removed from the surface of a 787 aircraft, would this reduce the air resistance? Does the seam on a baseball increase the amount of curvature when a curve ball is thrown? The answer to both is yes, and this behavior can be captured conceptually using a surface-air coupling constant, η. This parameter represents the degree to which the air adjacent to a surface moves with that surface and ranges from zero to one. Representative values for this constant are shown below with the measured result of 0.4 seeming to best fit the data that was measured.

Surface	***Air Coupling Constant (η)***
Frictionless Surface	0
Typical Surface	0.3
Measured Result	0.4
Sandpaper Surface	0.7
Maximum Possible	1

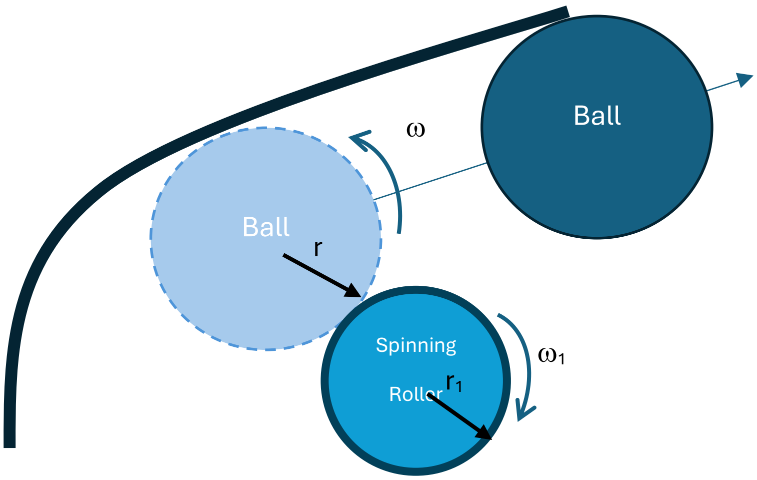
To visualize this, consider the air molecules in layers near a surface. The layer closest to the surface interacts most strongly with it. If the coupling constant is one, these air molecules move with the surface, representing maximum tangential interaction. If the constant is zero, they slide past the surface with no tangential interaction. In practice, the behavior lies between these extremes, with outer layers of air being progressively less influenced.

Consider the examples in the following table. For academic understanding, the first is infinite wall being struck with a column of air at angle θ. The second case is a sphere, which is the application for this paper.

Force	**Infinite Wall**	**Sphere**	**Impact on this Paper**
Drag	$f_{drag} \propto \sin^2\theta$	η =0: $f_{drag} \approx \frac{1}{3} \cdot f_{max}$ η =0.3: $f_{drag} \approx f_{max}$ η =1: $f_{drag} \approx \frac{2}{3} \cdot f_{max}$	The drag coefficient, C_d, already builds a coupling factor of about 0.4 into it.
Shear	$f_{shear} \propto \eta \cdot cos\theta$	f_{shear} decreases as η decreases, but not in a linear fashion.	Although Navier–Stokes assumes η=1 and lower η reduces shear, the relationship is nonlinear, so the classical model is used.
Magnus	0	$f_{Magnus} \propto \eta$	The Magnus force derivation needs to account for this coupling factor.

In the case of an infinite wall, the angle of incidence (θ) determines what is drag and what is shear force. For the drag term, the force is independent of the coupling coefficient and the sin term is squared because of the projection of the force back to the direction of motion. For the case of the sphere, a portion drag from direct interaction with the surface, but a larger portion of the drag is caused by the forming of a low pressure wake behind the sphere that slows it down.

Appendix I Ball Speed and Spin Induced by Spinning Roller



Symbol	*Parameter*	*Units*
v_0	Ball initial velocity	m/s
ω	Ball rotational velocity	rad/s
r	Ball radius	m
v_1	Spinning roller tangential velocity	m/s
ω_1	Spinning roller rotational velocity	rad/s
r_1	Spinning roller radius	m
k_1	Spinning roller to ball slip factor (ranges 0 to 1)	n/a
k_2	Ball to wall slip factor (ranges 0 to 1)	n/a

The tangential velocity of the spinning roller is:

$$v_1 = \omega_1 \cdot r_1 \qquad (I1)$$

Accounting for a slip factor, k1 (ranges 0 to 1), the tangential energy transferred to the ball is:

$$k_1 \cdot v_1 = \omega \cdot r + v_0 \qquad (I2)$$

If we assume that the ball rolls against the edge of the inside of the turret, then

$$k_2 \cdot v_0 = \omega \cdot r \qquad (I3)$$

Substituting (I3) into (I2) yields:

$$v_0 = \frac{k_1}{1 + k_2} \cdot v_1 \qquad (I4)$$

$$\omega = \frac{k_1 \cdot k_2}{1 + k_2} \cdot \frac{v_1}{r} \qquad (I5)$$

In the case that there is no slipping:

$$k_1 = k_2 = 1 \tag{I6}$$

Applying (I6) to (I4) and (I5) yields the following result.

$$v_0 = \frac{v_1}{2} \tag{I7}$$

$$\omega = \frac{v_1}{2 \cdot r} \tag{I8}$$

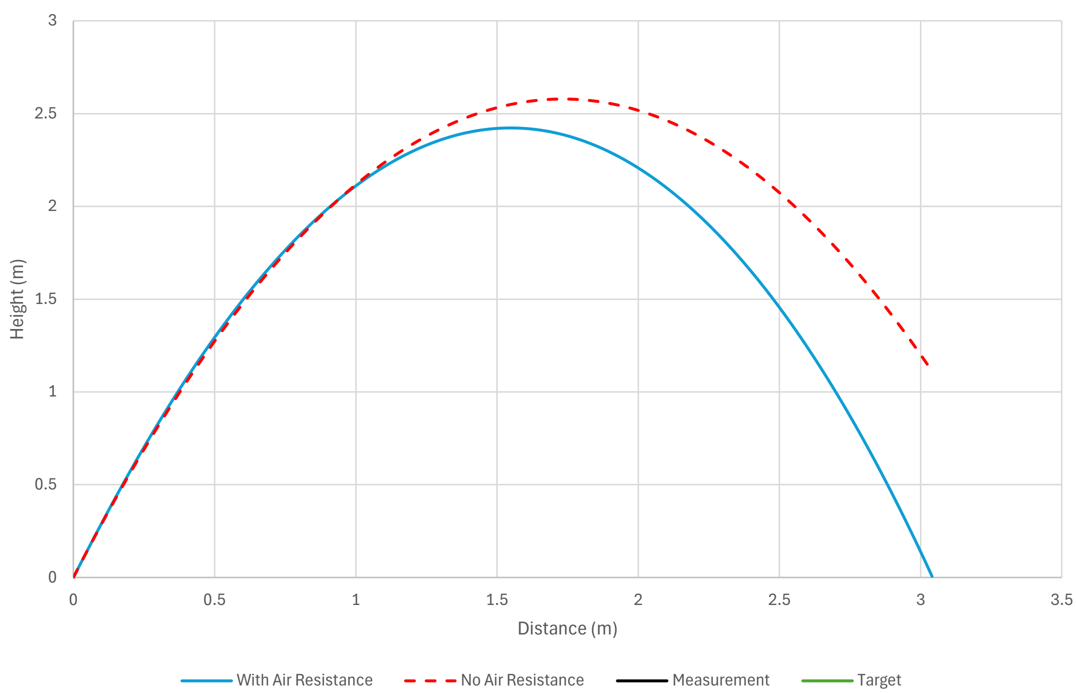
In situations where there is no slipping and one is trying to estimate v0 based on a measured trajectory, (I7) and (I8) can be combined to (I9) in order to get an idea of the rotational velocity of the ball relative to the projectile initial velocity.

$$\omega = \frac{v_0}{r} \tag{I9}$$

Appendix J Stationary Example with Air Resistance

Parameter	Symbol	Value	Unit
Initial Velocity	v_0	7.5	m/s
Angle	θ	65.5	deg
Acceleration Due to Gravity	g	9.8	m/s^2
Mass of Ball	m	0.23	kg
Quadratic Drag Term	α	5.775E-03	$N\cdot(s/m)^2$
Linear Drag Term	β	2.545E-05	N·s/m
Magnus Force Drag Term	γ	6.664E-04	$N\cdot s^2/m$
Rotational velocity	ω	15	rad/s
Target Distance	d	3	m
Target Height	h	1.2	m

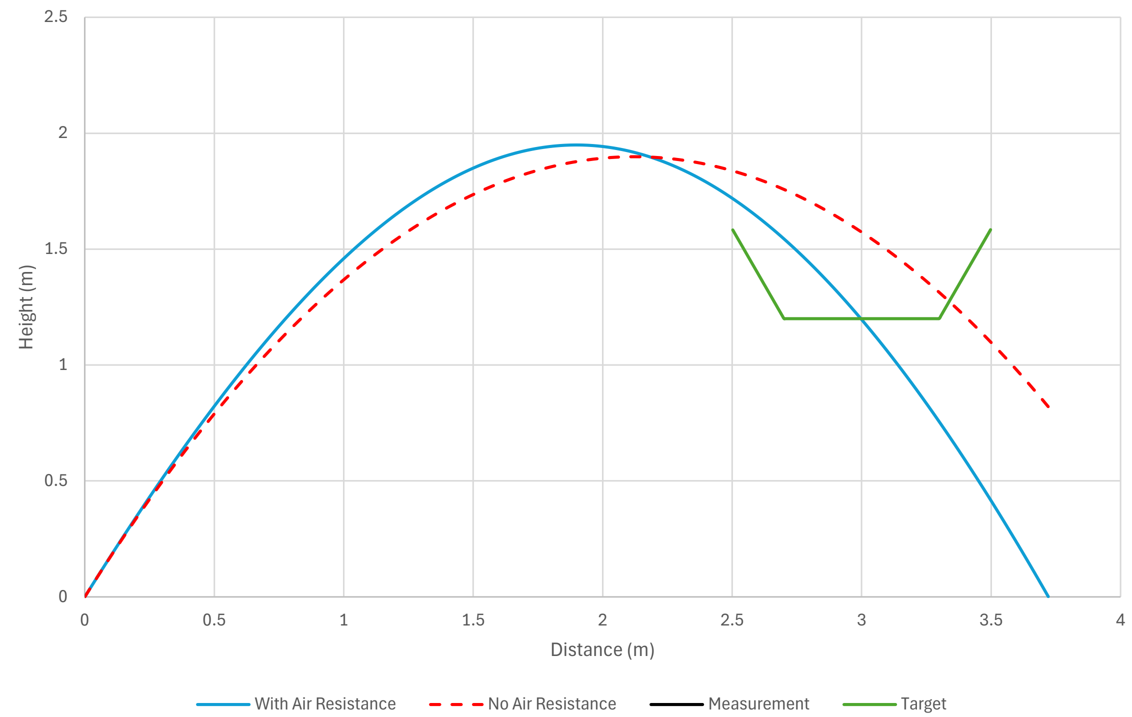
Parameter	Value	Unit
Peak Time	0.6964	s
Peak Height	2.3764	m
Time to Target	0.9646	s
Time to Hit Ground	1.3928	s
Range	4.3319	m



Appendix K Example for Moving Turret with Air Resistance

Parameter	Symbol	Value	Unit
Initial Velocity	v_0	6.4	m/s
Launch Angle	θ	72.4	deg
Acceleration Due to Gravity	g	9.8	m/s^2
Target Distance	d	3	m
Target Height	h	1.2	m
Azimuth Angle	ϕ	-30.0000	deg
Vehicle Heading	ϕ_m	30.0000	deg
Vehicle Speed	v_m	2.0000	m/s
Mass of Ball	m	0.23	kg
Rotational velocity	$\omega / (2\pi)$	15	rev/s
Spin Decay Coefficient	τ	7.00	s
Quadratic Drag Term	α	5.78E-03	$N\cdot(s/m)^2$
Linear Drag Term	β	2.54E-05	$N\cdot(s/m)$
Magnus Force Drag Term	γ	6.664E-4	$N\cdot(s^2/m)$

XY Plane Graph



XZ Plane Graph